THE DISTRIBUTION OF TANGENTIAL STRESSES IN AN INCOMPRESSIBLE TURBULENT BOUNDARY LAYER

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We investigate an isothermal turbulent boundary layer with positive pressure gradient. We obtain results on the distribution of tangential stresses in the layer which are approximated by a simple computation equation.

It was shown in [1-3] that the familiar semiempirical methods of calculating the characteristics of the turbulent flow of an incompressible liquid when there is a positive pressure gradient give contradictory results and agree poorly with the results of measurements obtained by using the most highly developed modern measuring apparatus. This is explained by the fact that the present computational methods are based on limited experimental material. Obtaining experimental data which is reliable in the structure and the outlet characteristics of turbulent flow involves great technical difficulties in measuring the initial quantities, processing the results and generalizing them.

In a well-known experimental paper Schubauer and Klebanoff [4] studied the turbulent boundary layer with a large model of the profile in a closed circuit wind tunnel. Experiments were carried out at a Reynolds number in the incident flow of $\mathrm{Re}_\theta = 18{,}700$, increasing along the contour of the profile to 77,000. All three components of the turbulent intensity and the correlation between two components of the fluctuating velocity at a point and between these components at different points were measured. The experimental results of this essentially unique paper have been used by many authors to develop methods for calculating the turbulent boundary layer and comparing the results of semiempirical methods with experiment. But the limitation of the measurements in these experiments to one Reynolds number for the incident flow and one law for the variation of the pressure gradient makes it difficult to establish a relation between the tangential and the normal stresses and the average characteristics of the flow. And without establishing a clear physical basis for this relation it is impossible to develop a deep understanding of the structure of the turbulence which is the most important prerequisite for constructing a valid theory and reliable methods of calculating the turbulent boundary layer. In flows with a positive pressure gradient the effect of the normal stresses on the structure of the boundary layer and its characteristics has no less value than the effect of the tangential stresses. As boundary layer breakaway is approached, the effect of the normal stresses becomes definitive.

For the future development of the theory and methods of calculating the turbulent boundary layer of a liquid it is necessary to accumulate experimental data obtained using modern laboratory apparatus.

In this paper we investigate an isothermal boundary layer in an air flow in axisymmetric and plane diffusors. The investigations were carried out in an open working section wind tunnel. The air velocity was smoothly controlled by two baffle plates. The experiments embraced a range of Reynolds numbers at the diffusor inlets from 48,500 to 202,000 (Table 1).

The axisymmetric diffusors had: inlet diameter 100 mm, length 500 mm, aperture angle 8 to 10°. The control cross sections at which the required variables were measured were at the following distances from the inlet cross section: 0, 30, 75, 135, 202, 280, and 360 mm.

The working section of the plane rectangular diffusor with inlet cross section 40 × 180 mm was of length 174 mm. The upper and lower walls were movable, which made it possible to change the aperture

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TABLE 1. Reynolds Numbers at the Inlet Sections of the Diffusors

Axisymmetric diffusor			Plane diffusor		
aperture angle, deg	Reynolds number	notation for points in Fig. 1	aperture angle, deg	Reynolds number	notation for points in Fig. 1
8	48500	*1*	10	58200	*7*
	135500	*2*			
	200000	*3*	12	57600	*8*
10	52700	*4*			
	145700	*5*	14	58800	*9*
	202000	*6*			

angle. The experiments were carried out at aperture angles of 10, 12, and 14°. In all cases the variables were measured at cross sections at distances of 0, 30, 60, 90, 130, and 170 mm from the inlet cross section.

At each section the mean velocity profile, the intensity of the turbulence of the longitudinal and normal velocity components, the profile of the turbulent tangential stress, and the correlation between the longitudinal and normal fluctuations of the velocity at a point were measured.

The measurements were made with an electrothermal anemometer of type UTA-5B, a detailed description of which is contained in [5]. The sensitive element of this instrument was a tungsten thread of diameter 11 μ. The length of the thread was established from the recommendation of [6, 7]. The electrothermomoanemometer transducers were calibrated from the readings of a Pitot tube attached to a micromanometer of type MMN. Two forms of transducer were used: with thread, perpendicular to the axis of the transducer and with thread at an angle of $\pi/4$ to the axis of the transducer. The transducer could be inserted in a plug in two positions differing by an angle of $\varphi = \pi$ with respect to the longitudinal axis. The length of the transducer was 300 mm; its body was a stainless steel tube of external diameter 4 mm. The tapered rods to which the thread was fixed were of stainless steel. They were of diameter 1 mm at the base and stood out from the transducer body a distance of 12 mm. The distance between the rods where the thread was attached was 2 mm.

A transducer of the first type was used to measure the mean velocity of the flow and the intensity of the turbulence of the longitudinal velocity component; a transducer of the second type was used to measure the mean velocity of the flow and the turbulent tangential stress. From the measurements made with the two transducers the intensity of the turbulence of the normal velocity component was calculated.

By generalizing the experimental results on the turbulent tangential stresses we can obtain reliable approximations for the distribution of the tangential stress in the boundary layer in flows with a positive pressure gradient.

Fedyaevskii [8] represented the distribution of the tangential stress $\tau(\eta)$ in the boundary layer as a fourth degree polynomial in powers of $\eta = y/\delta$. To determine the unknown constants in this polynomial he used the following boundary conditions from the momentum equation for a boundary layer:

$$\tau = \tau_w; \quad \frac{\partial \tau}{\partial \eta} = -\rho u_1 \delta \frac{du_1}{dx}; \quad \frac{\partial^2 \tau}{\partial \eta^2} = 0 \quad \text{for} \quad \eta = 0,$$
$$\tau = \frac{\partial \tau}{\partial \eta} = 0 \quad \text{for} \quad \eta = 1. \tag{1}$$

The tangential stress distribution satisfying (1) has the form

$$\frac{\tau}{\rho u_1^2} = \frac{\tau_w}{\rho u_1^2}(1 - 4\eta^3 + 3\eta^4) - \frac{\delta}{u_1} \cdot \frac{du_1}{dx}(\eta - 3\eta^3 + 2\eta^4), \tag{2}$$

where

$$-\frac{\delta}{u_1} \cdot \frac{du_1}{dx} = \frac{\delta}{\rho u_1^2} \cdot \frac{dp}{dx}.$$

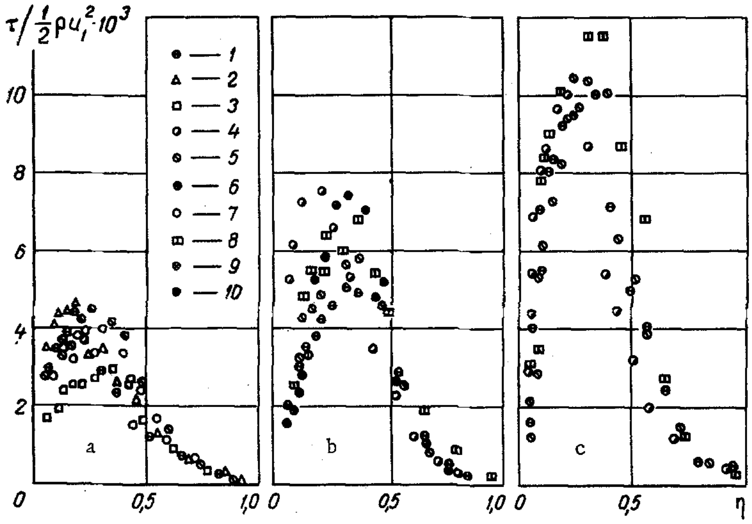

Fig. 1. Distribution of the tangential stresses at cross sections of the boundary layer for various values of the form parameter H from the data of [5] [a) H = 1.5-1.53; b) H = 1.73-1.75; c) H = 1.92-1.98]. For the values of the points 1-9 cf. Table 1, of points 10, cf. [4]; points 1-10 are also used in Figs. 2 and 3.

Comparison of the distribution $\tau(x)$ computed from (2) with experiment using the results of [4] reveals a large discrepancy. We see this in Fig. 4, in which we compare the computed values of the tangential stress with its experimental values from the data of [4] at three cross sections of the boundary layer: before the start of the flow with $dp/dx > 0$ (5.3 mm), before detachment of the boundary layer (7.6 mm) and at an intermediate cross section (6.86 mm). At the second and third cross sections of the boundary layer $(2\delta/u_1)\cdot(du_1/dx)$ is almost the same, but the form parameter H at the third cross section is much greater than at the second. We see that the tangential stress distributions are significantly different at these cross sections. Our experimental results also indicate the relation between the tangential stress distribution in the layer and the form parameter H. Figure 1 shows the distribution of $\tau/(\rho u_1^2/2)$ for three values of H. The tangential stress has different values at cross sections with practically the same value of $(2\delta/u_1)(du_1/dx)$. For example, in an axisymmetric diffusor with aperture angle of 8° and a Reynolds number of $4.85\cdot 10^4$ in the incident flow we have:

$$\text{for}\quad H=1.51\quad \frac{2\delta}{u_1}\cdot\frac{du_1}{dx}=-0.0326;\quad \frac{\tau}{\frac{1}{2}\rho u_1^2}=4.3\cdot 10^{-3};$$

$$\text{for}\quad H=1.75\quad \frac{2\delta}{u_1}\cdot\frac{du_1}{dx}=-0.0298;\quad \frac{\tau}{\frac{1}{2}\rho u_1^2}=5\cdot 10^{-3};$$

$$\text{for}\quad H=1.98\quad \frac{2\delta}{u_1}\cdot\frac{du_1}{dx}=-0.0324;\quad \frac{\tau}{\frac{1}{2}\rho u_1^2}=10\cdot 10^{-3}.$$

In a plane diffusor with an aperture angle of 12° and a Reynolds number of $5.76\cdot 10^4$ in the incident flow we obtain:

$$\text{for}\quad H=1.73\quad \frac{2\delta}{u_1}\cdot\frac{du_1}{dx}=-0.0502;\quad \frac{\tau}{\frac{1}{2}\rho u_1^2}=6.18\cdot 10^{-3};$$

$$\text{for}\quad H=1.90\quad \frac{2\delta}{u_1}\cdot\frac{du_1}{dx}=-0.0554;\quad \frac{\tau}{\frac{1}{2}\rho u_1^2}=10.64\cdot 10^{-3}.$$

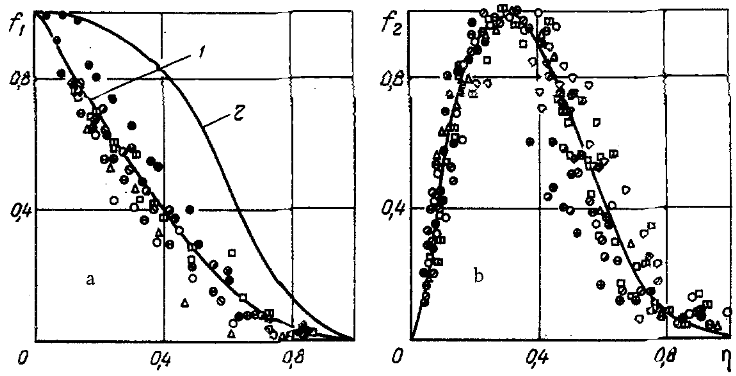

Fig. 2. Graph of the function f_1 (a) [1) from Eq. (5); 2) from (2)] and the function f_2 (b) (the notation for the points is as in Fig. 1).

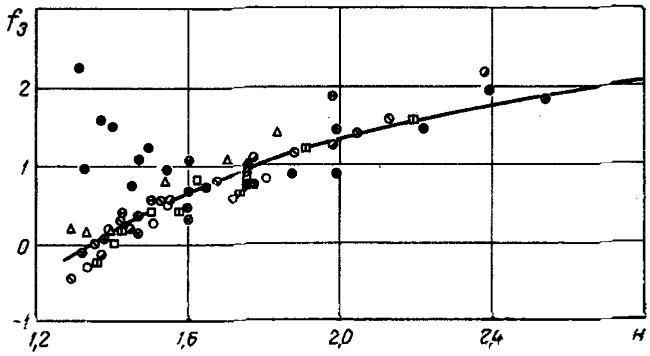

Fig. 3. Graph of the function f_3 (the notation for the points is as in Fig. 1).

Evidently, Eq. (2) can be made more precise if we take into account the relation between τ and H. This can be done on the basis of the experimental results obtained in our paper by writing (2) as

$$\frac{\tau}{\rho u_1^2} = \frac{\tau_w}{\rho u_1^2} f_1(\eta) - \frac{\theta}{u_1}\cdot\frac{du_1}{dx} f_2(\eta) f_3(H), \tag{3}$$

where $f_1(\eta)$ is a function defining the tangential stress distribution in the absence of a pressure gradient; $f_2(\eta)$ is a function expressing the tangential stress distribution for fixed values of the pressure gradient $(\theta/u_1)(du_1/dx)$ and the form parameter H; $f_3(H)$ is a function taking account of the effect of the form parameter on the tangential stress distribution.

In Eq. (3) we have introduced the momentum thickness θ instead of the boundary layer thickness δ in Eq. (2), since θ can be determined more exactly than δ.

We established the forms of the functions $f_1(\eta)$, $f_2(\eta)$, and $f_3(H)$ by approximating the experimental results obtained by studying the boundary layer in axisymmetric and plane diffusors.

Figure 2a shows the tangential stress distribution at initial cross sections of diffusors from the results of measurements. The pressure gradient of these cross sections was close to zero; the value of H was found to be 1.33; the friction coefficient was determined from the familiar Ludwieg–Tillmann equation

$$c_f = 0.246\left(\frac{u_1\theta}{\nu}\right)^{-0.268}\cdot 10^{-0.678H}. \tag{4}$$

The measured tangential stress in the boundary layer is well approximated by the equation

$$f_1(\eta) = 1 - 2.1\eta^{1.1} + 1.1\eta^{2.1}, \tag{5}$$

satisfying the boundary conditions: $f_1(0) = 1$; $f_1'(0) = 0$; $f_1(1) = 0$; $f_1'(1) = 0$ (primes denote differentiation with respect to η). Figure 2a also shows a curve constructed from (2). To determine the form of the function $f_2(\eta)$ from the results of the measurements we constructed a graph of the ratio of the local tangential stress

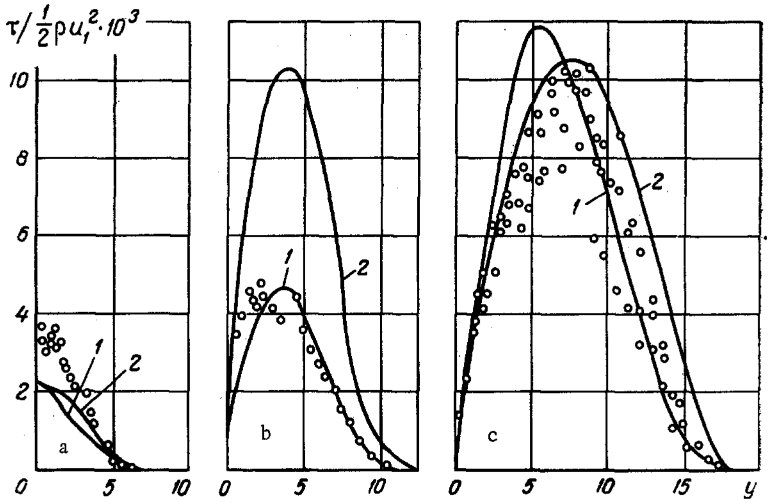

Fig. 4. Tangential stress distributions at cross sections of the boundary layer: a) at the start of the flow $dp/dx \geq 0$ ($x = 5.3$ mm, $c_f = 0.00207$; $(2\delta/u_1)(du_1/dx) = 0$, $H = 1.35$); b) at an intermediate cross section ($x = 6.86$ mm, $c_f = 0.00115$; $(2\delta/u_1)(du_1/dx) = -0.0391$; $H = 1.60$); c) near breakaway ($x = 7.6$ mm, $c_f = 0.00038$; $(2\delta/u_1)(du_1/dx) = -0.0395$; $H = 2.22$); 1) from (3); 2) from (2).

τ to the maximum value τ_{max} at the given cross section as a function of the nondimensional coordinate η. The graph shows τ/τ_{max} for all values of the boundary layer when $H > 1.5$. From this construction it appears that the experimental values of τ/τ_{max} at sections with different pressure gradients and form parameters H in the range of Reynolds numbers considered lie on a curve with a maximum at $\eta \approx 0.3$ (Fig. 2b), with an acceptable scatter. The curve $f_2(\eta)$ satisfying the boundary conditions: $f_2(0) = 0$; $f_2(1) = 0$; $f_2'(1) = 0$; $f_2(0.3) = 1$; $f_2'(0.3) = 0$ (primes denote differentiation with respect to η) is given by the equation

$$\begin{aligned} f_2(\eta) &= \frac{20}{3} - \frac{100}{9}\eta^2 && \text{for } 0 \leqslant \eta \leqslant 0.3, \\ f_2(\eta) &= 11.69(1-\eta)^3 - 12.53(1-\eta)^4 && \text{for } 0.3 \leqslant \eta \leqslant 1. \end{aligned} \tag{6}$$

The function $f_3(H)$ can be expressed by rewriting Eq. (3) and noting that when $H > 1.5$, $f_2(0.3) = 1$:

$$f_3(H) = \frac{1}{2}\left[\frac{\dfrac{\tau}{\frac{1}{2}\rho u_1^2} - c_f f_1(0.3)}{-\dfrac{\theta}{u_1}\cdot\dfrac{du_1}{dx}}\right]. \tag{7}$$

The graph of $f_3(H)$ is given in Fig. 3. In constructing the graph the friction coefficients was determined from (4). All the other variables on the right side of (7) were obtained at each cross section of the boundary layer for the appropriate value of H from the results of the measurements. We see that the experimental points are well grouped about the curve which can be described by the equation

$$f_3(H) = \frac{H - 1.33}{0.25H}, \tag{8}$$

where 1.33 is the value of H when the velocity distribution follows a power law with exponent 1/6.

Figure 3 also shows the experimental points from [4]. When $H < 1.5$ they lie above the approximating curve, which, evidently is explained by the small values of $(\theta/u_1)(du_1/dx)$ in that region of H.

Comparison of the computed tangential stress distribution, using Eqs. (2) and (3) with the experimental distributions from [4] is made in Fig. 4 for three cross sections of the boundary layer with positive pressure gradient. A significant divergence between the computational results using Eq. (3) and the experimental results is observed at the point where the positive pressure gradient begins, where the first term on the right side of Eq. (3) (for H = 1.35, $(\theta/u_1)(du_1/dx) \to 0$) makes a significant contribution to τ. The value of $2\tau/\rho u_1^2$ for this cross section, computed from (4) using experimental values of H and Re_θ from [4] is almost half the corresponding experimental value of the tangential stress obtained by the authors [4] from direct measurement. For example, when $Re_\theta = 18{,}700$, H = 1.35, from (4) the friction coefficient is $2.13 \cdot 10^{-3}$, while from direct measurements it is $(3.3\text{-}3.7) \cdot 10^{-3}$. Thus we can conclude that the experimental points in Fig. 4 yield an overestimate of the tangential stress distribution.

At an intermediate cross section (6.86 mm) good agreement was observed between the computed and experimental values of τ. At a third cross section (7.6 mm) the computed curve diverges from the experimental points in the upper left part of the graph. But, as analysis of the results of [4] shows, the experimental points of Schubauer and Klebanoff give a maximum value of the tangential stress which moves in the direction of larger values of η as the boundary layer develops. As H varies from 1.6 to 2.39, the value of the coordinate η at which $\tau/(\rho u_1^2/2)$ has its maximum varies from 0.15 to 0.4. At a cross section 7.6 mm from the start of the motion with $dp/dx > 0$, the maximum of $\tau/(\rho u_1^2/2)$ occurs for $\eta = 0.4$. A movement of the maximum is difficult to justify physically. The maximum tangential stress, calculated from (3) for all H > 1.5, is at $\eta = 0.3$. Fedyaevskii [8] gave a function f_2 which has a maximum at $\eta = 0.4$. Hence there is some disagreement between the left part of the curve and the experimental points. The right part gives good agreement with experiment.

NOTATION

x	is the distance along the diffusor axis, measured in the direction of the flow, mm;
y	is the distance along the normal to the diffusor axis, measured from the surface, mm;
u_1	is the mean velocity at the edge of the boundary layer, m/sec;
$c_r = \tau/(\rho u_1^2/2)$	is the turbulent friction stress;
δ	is the boundary layer thickness;
δ^*	is the displacement thickness;
θ	is the momentum thickness;
$H = \delta^*/\theta$	is the form parameter of the velocity profile in the boundary layer;
Re_θ	is the Reynolds number computed from the momentum thickness;
ν	is the kinematic viscosity, m^2/sec.

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